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Subsets of an interval whose product is a power

Paul Erdős, Janice L. Malouf, J.L. Selfridge*, Esther Szekeres

Northern University, DeKalb, IL 60201, USA

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Dedication

The surviving authors gratefully dedicate this paper to the memory of Paul Erdős. He was a close friend to each of us for practically our entire mathematical careers. We hope that he is reading The Book.

Abstract

We form squares from the product of integers in a short interval $[n, n+t_n]$, where we include n in the product. If p is prime, $p|n$, and $\binom{p}{2} > n$, we prove that p is the minimum t_n. If no such prime exists, we prove $t_n \leqslant \sqrt{5n}$ when $n > 32$. If $n = p(2p-1)$ and both p and $2p+1$ are primes, then $t_n = 3p > 3\sqrt{n/2}$. For $n(n+u)$ a square $> n^2$, we conjecture that a and b exist where $n < a < b < n+u$ and nab is a square (except $n=8$ and $n=392$). Let $g_2(n)$ be minimal such that a square can be formed as the product of distinct integers from $[n, g_2(n)]$ so that no pair of consecutive integers is omitted. We prove that $g_2(n) \leqslant 3n-3$, and list or conjecture the values of $g_2(n)$ for all n. We describe the generalization to kth powers and conjecture the values for large n. © 1999 Elsevier Science B.V. All rights reserved

1. Introduction

A result of Erdős and Selfridge [3] states that the product of consecutive integers is never a power (see also [4]). We investigate related problems of forming powers from the product of integers in an interval, but with conditions that allow for gaps in the interval. The statements of the main results of Section 2 appeared in the solution of advanced problem #6655 in the *Monthly* [1].

We start with a natural number n that is not a kth power, and select a set of integers larger than n whose product with n forms a kth power. We seek the minimal interval in which this can be done. We may also seek to do this subject to the condition that we never omit k consecutive numbers, i.e. that the largest gap between integers used is never greater than k. In the case of squares, this means that we may omit a number $n+i$ from the product provided that both $n+i-1$ and $n+i+1$ are included. The

* Corresponding author. E-mail: selfridg@math.niu.edu.

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unrestricted problem with squares is to start with n and while allowing arbitrarily large gaps, to form a square as such a product, minimizing the largest number used. We first consider the problem with unrestricted gap size.

Notation: The letter p is reserved to represent primes.

2. Squares without gap condition

Given n, not a square, we seek the minimal value t_n such that the product of n together with a subset of $(n, n+t_n]$ is a square.

Write $n = ap$ where p is the largest prime factor of the square-free part of n. If $p > 2a + 1$ we obtain 6 integer products which are products of a, $a+\frac{1}{2}$, $a+1$ and $p-1$, p, $p+1$. We get $a \cdot p$, $a(p+1)$, $(a+1/2)(p-1)$, $(a+\frac{1}{2})(p+1)$, $(a+1)(p-1)$, $(a+1)p$. The condition $p > 2a+1$ ensures that all factors lie in $[n, n+p]$. For example, $44 \cdot 48 \cdot 45 \cdot 54 \cdot 50 \cdot 55$ shows that $t_{44} = 11$. This value of t_n is clearly best possible, since p divides the square-free part of n. This construction (called a 6-pack) also works when p is replaced by an odd composite number, but then usually is not best possible.

From now on, we consider only values of n for which $p \leqslant 2a + 1$. Define the real number $x > 0$ by $x(2x-1) = n$. Note that $p = 2a + 1$ implies that $p = 2x$.

Lemma 1 (Walk Lemma). *Given any two distinct positive integers b_0 and b_k, with $x < b_0$, $b_k \leqslant 2x$, we can choose integers from $(n, n+3x]$ whose product is $b_0 b_k$ times a square.*

Notation. The numbers chosen in $(n, n+3x]$ are denoted by $a_1, \ldots, a_k$, where $a_i = b_{i-1} b_i$. Finding appropriate numbers b_i given b_0 and b_k is called *walking* from b_0 to b_k.

Proof. There are two cases according as $b_0 b_k \leqslant n$ or $b_0 b_k > n + 3x$. If $n < b_0 b_k \leqslant n + 3x$, the lemma is satisfied by the integer $b_0 b_k$.

If $b_0 b_k \leqslant n$, we walk from the smaller to the larger ($b_0 < b_k$). We choose for b_1 the smallest integer so that $b_0 b_1 > n$ and we set $a_1 = b_0 b_1$ and find b_2 using an *upstep*. An upstep means that we take b_i to be $b_{i-2} + 1$. If here, or at any point, $b_i = b_k$, then we set $k = i$, $a_1, \ldots, a_k = b_0 b_1^2 \ldots b_{k-1}^2 b_k$, and the walk is finished. Otherwise we find b_3 using a *downstep* and alternate upsteps and downsteps until $b_i = b_k$. There are two kinds of downstep. If $b_{i-2} - 1 = b_k$, or if $b_{i-1}(b_{i-2} - 2) \leqslant n$, then we set $b_i = b_{i-2} - 1$, and we have a *single downstep*. Otherwise, we set $b_i = b_{i-2} - 2$, a *double downstep*.

If $b_0 b_k > n + 3x$, we walk from the larger to the smaller ($b_0 > b_k$). We choose b_1 to be the largest integer such that $b_0 b_1 \leqslant n + 3x$, set $a_1 = b_0 b_1$, and start with a downstep and then alternate as in the first case until $b_i = b_k$.

We show using elementary convexity arguments that the upsteps and downsteps never take us out of the interval $(n, n+3x]$.

Proof of first case: We walk from b_0 to b_k with $b_0 < b_k$. We show that the first upstep does not take us above the bound $n+3x$. We may assume the worst case, when n is an integer multiple of b_0, say $n = b_0 h$. The walk starts with $a_1 = (h+1)b_0$, then takes an upstep giving $a_2 = (h+1)(b_0+1)$, (to be followed by a downstep). We show that $hb_0 + h + b_0 + 1 < n + 3x$. From $hb_0 = n = x(2x-1)$ and $x < b_0 < b_k \leqslant h < 2x - 1$, we get $h + b_0 < x + (2x-1)$, and hence $hb_0 + h + b_0 + 1 = n + h + b_0 + 1 < n + x + (2x-1) + 1 = n + 3x$.

More generally, this argument shows that any upstep which follows a downstep will never exceed $n+3x$. After an upstep, a downstep can always be made since the net change from an upstep followed by a single downstep is positive.

Proof of second case: It suffices to show that the initial downstep does not take us below the interval $(n, n+3x]$. The argument for other downsteps and the upsteps is as above.

We walk from $b_0 \leqslant 2x$ to the smaller number b_k. We show that the downstep from a_1 (which will subtract a multiple of b_0) results in an integer $a_2 > n$.

Define y by $b_0 y = n + 3x = 2x^2 + 2x$ $(= 2x(x+1))$. Then $y \geqslant x + 1$, and if y is an integer, $a_1 = n + 3x$, and a downstep can easily be made. Assume y is not an integer. Then $b_0(\lfloor y \rfloor)$ is the largest multiple of b_0 which is $\leqslant n + 3x$, and thus is equal to a_1. We need to check that $(b_0 - 1)(\lfloor y \rfloor) > n$. Since $x + 1 \leqslant y < b_k < b_0 \leqslant 2x$ with $b_0 y = 2x(x+1)$, we have $b_0 + y \leqslant (x+1) + 2x = 3x + 1$, hence $(b_0 - 1)(\lfloor y \rfloor) > (b_0 - 1)(y - 1) = b_0 y + 1 - b_0 - y \geqslant (2x^2 + 2x) + 1 - 3x - 1 = n$. $\square$

Theorem 2. *If $p \leqslant 2a + 1$, with a, p, x defined as above, then $t_n \leqslant 3x$, except when $n \in \{2, 3, 8, 10, 32\}$.*

Proof. The strategy is to write the square-free part of n times a square as the product of two numbers r, s or four numbers r, s, u, v in the interval $(x, 2x]$. Form r as a product of primes dividing the square-free part of n by beginning with p (the largest) and multiplying by the next largest ones as long as r remains $\leqslant 2x$. Put $s = u = 1$, and if a prime q makes r too large, s is multiplied by q and further primes, keeping $s \leqslant 2x$, and if necessary u is multiplied by any remaining primes. Continue until rsu is equal to the square-free part of n. (Since $2x > \sqrt{n}$ and the largest prime q dividing s satisfies $rq > 2x > \sqrt{n}$, u will certainly suffice to contain all remaining prime factors). We multiply by a power of 2, if needed, to arrange that each of r, s (and u if $u > 1$) lies in the interval $(x, 2x]$.

If the number of 'extra 2's' used was even and $u = 1$, then r and s alone contain all prime factors, and we apply Lemma 1 using r and s.

Otherwise we make two walks. If $u = 1$, multiply it by 2's so that $u \in (x, 2x]$. Now define a fourth number $v \in (x, 2x]$, where v is either a square or twice a square, according as the number of extra 2's used to make the values of r, s, and u large enough is even or odd, and then produce two walks between disjoint pairs of r, s, u, v. The product of all numbers in these walks has the same square-free part as does n.

Table 1
Factors used with $n \in [120, 153]$, $p \leqslant 2a+1$

n	r	s	u	v
125	15	12	—	—
128	10	12	15	9
136	17	10	15	12
147	12	16	—	—

Table 2
Products for t_n for $n \in [2, 35]$, $p \leqslant 2a+1$

n	t_n	$3x$	Description of product		
2	4	3.84	$2\times 3\times 2\cdot 3$		$p(p-1)$, $t_n=2p$
3	5	4.5	$3\times 2\cdot 3\times 2\cdot 4$		
6	6	6	$2\cdot 3\times 2\cdot 4\times 3\cdot 4$		$p(p-1)$, $t_n=2p$
8	7	6.80	$2\cdot 4\times 2\cdot 5\times 3\cdot 4\times 3\cdot 5$		
10	8	7.5	$2\cdot 5\times 2\cdot 6\times 3\cdot 5\times 3\cdot 6$		
12	8	>8	$3\cdot 4\times 3\cdot 5\times 4\cdot 5$		
15	9	9	$3\cdot 5\times 3\cdot 6\times 4\cdot 5\times 4\cdot 6$	$x(2x-1)$	x, $2x+1$ primes
18	9	>9	$3\cdot 6\times 4\cdot 6\times 3\cdot 9$		Omit $4\cdot 9$
20	10	>10	$4\cdot 5\times 4\cdot 6\times 5\cdot 6$		$p(p-1)$, $t_n=2p$
21	7	10.5	$3\cdot 7\times 3\cdot 9\times 4\cdot 7$		Omit $4\cdot 9$
24	8	>11	$3\cdot 8\times 3\cdot 9\times 4\cdot 8$		Omit $4\cdot 9$
27	8	>11	$3\cdot 9\times 4\cdot 7\times 2\cdot 3\cdot 5\times 2\cdot 4\cdot 4\times 5\cdot 7$		Omit $4\cdot 9$ (6-pack)
28	12	12	$4\cdot 7\times 4\cdot 8\times 5\cdot 7\times 5\cdot 8$		
30	12	>12	$5\cdot 6\times 5\cdot 7\times 6\cdot 7$		
32	13	12.8	$4\cdot 8\times 5\cdot 8\times 5\cdot 9$		Omit $4\cdot 9$
35	13	>13	$5\cdot 7\times 5\cdot 8\times 6\cdot 7\times 6\cdot 8$		

This can be done if there is a square and twice a square in $(x, 2x]$. Note that if $4.5 \leqslant x < 8$, then 8 and 9 are available. Also, when $x \geqslant 9$, there is a square and twice a square in $(x, 2x]$. Whenever $n = d_1 d_2$ and $d_2 \leqslant 2d_1 - 1$ then we can walk from d_1 to d_2 ($t_n \leqslant d_1 + d_2 + 1 \leqslant 3x$).

With $8 \leqslant x < 9$, there remain only 125, 128, 136, and 147. These can be handled by walking from r to s, and if necessary from u to v, as given in Table 1. For example when $n = 128$, $n + 3x < 153$ and we walk from 10 to 12 and from 15 to 9. Since $10 \cdot 12 < 128$, we start with $a_1 = 130 = 10 \cdot 13 = b_0 b_1$. Continuing, $a_2 = 143 = 13 \cdot 11 = b_1 b_2$. The walk finishes with $a_3 = 132 = 11 \cdot 12 = b_2 b_3$. The walk from 15 to 9 is trivial, so $128 \times 10 \cdot 13^2 11^2 12 \cdot 15 \cdot 9$ is a square and $t_{128} \leqslant 15$. A straightforward exercise shows that $t_{128} = 15$ and that 130, 143, 132, 135 are necessary factors. When $n = 147$, the walk from 16 to 12 gives the 6-pack $7 \cdot 21, 7 \cdot 22, 20 \cdot 15/2, 22 \cdot 15/2, 8 \cdot 20, 8 \cdot 21$ and $t_{147} \leqslant 21$. In fact $147 \cdot 150 \cdot 162$ is a square and $t_{147} = 15$.

Only the cases with $n < 36$ ($x < 4.5$) and $p \leqslant 2a+1$ remain. These are in Table 2 where a product showing t_n is given. The reader can easily confirm that the value claimed is indeed t_n (i.e. is minimal). We illustrate $t_{32} = 13$.

If we suppose that $32+t_{32}$ is smaller that 45, the value in Table 2 for $32+t_{32}$, then we must use a number with 2 in the square-free part. We cannot use 34 (or 38) since 51 (or 57) exceeds 45. Use of 40 forces 35, then 42, and the parity of the power of 2 is unchanged. Conversely, 42 forces 35 and 40, so there is no usable number with 2 in its square-free part, a contradiction. □

Remarks on the bounds. When $n=(p-1)p$, since another multiple of p is needed to form a square, we must go as far as $p(p+1)$. Thus the solution $n\cdot(p-1)(p+1)\cdot p(p+1)$ is best possible for this infinite set of values of n with $p\leqslant 2a+1$. Note that $p(p+1)=n+2p>n+2\sqrt{n}+1$.

Consider the (doubtless infinite) set of $n=p(2p-1)$ where p and $2p+1$ are both prime. Here $t_n=3p$, the product being $p(2p-1)\cdot p(2p)\cdot(p+1)(2p-1)\cdot 2p(p+1)$. Since $x=p$, we have $t_n=3x$, demonstrating that the bound given in Theorem 2 is exact. Notice that $3x$ approaches $c\sqrt{n}$, where $c=3/\sqrt{2}\sim 2.1213$.

3. Squares with restricted number of factors

In this section we make a square product from $[n, n+u_n(f)]$ where $u_n(f)$ is minimal such that the product has at most f factors including n. We write $n=rs^2$, with $r>1$ and square-free. It is immediate that $n+u_n(2)=r(s+1)^2$, and thus $u_n(2)=r(2s+1)>2.828\sqrt{n}$. In particular for square-free n, $u_n(2)=3n$.

We first show that $u_n(4)<u_n(2)$ and thus $t_n<u_n(2)$. This is easy since $rs^2\cdot(rs+1)s\cdot rs(s+1)\cdot(rs+1)(s+1)$ is a square and $(rs+1)(s+1)<(rs+r)(s+1)=r(s+1)^2$.

For $n=p(p-1)$, we noted above that $t_n=u_n(3)=2p$. When $n=ap$ with $p=4a+1$, $ap\cdot 4a(a+1)\cdot(a+1)p$ shows that $t_n=u_n(3)=p$, and similarly when $p=4a+3$.

So far, none of our constructions yield $t_n<c\sqrt{n}$. Here are two families which have $t_n=u_n(3)=c\sqrt[3]{n}$. To generalize $48\cdot 50\cdot 54$, put $n=k^6-4k^2$ and $u=k^2+2$. Then $n(n+u-4)(n+u)$ is square. Better, $n=4k^6-4k^2$ and $u=k^2+1$ yields $n(n+u-2)(n+u)$ square, generalizing $240\cdot 243\cdot 245$.

In these examples, $t_n=u_n(3)$. The question naturally arises, for which non-squares is $u_n(3)=u_n(2)$? We will return to this, but we first note that $t_n=u_n(4)$ for the family given in the final paragraph of the preceding section, namely, $n=p(2p-1)$ with p and $2p+1$ both prime, and that $u_n(3)$ is almost always larger than $u_n(4)$ for this family. In the remainder of this section we investigate the following conjecture.

Conjecture 3 (The 392 Problem). When $n>8$, $u_n(3)<u_n(2)$, except for $n=392$.

We first show that if $n=rs^2$ and $r=r_1r_2$, with both $r_1, r_2<r$, then $n+u_n(3)<r(s+1)^2$.

Theorem 4. *When r is not prime, $u_n(3)<u_n(2)$.*

Proof. Let $r_1x^2 = r_1r_2s^2$. Now $(2x+1) = 2s\sqrt{r_2} + 1 < 2sr_2 + r_2 = r_2(2s+1)$. Thus $r_1(x+1)^2 = r_1x^2 + r_1(2x+1) < rs^2 + r(2s+1) = r(s+1)^2$. This shows that $r_1\lfloor x+1\rfloor^2$ is in the interior of the interval. Similarly for r_2. □

We now may assume that r is a prime, and that $n = ps^2$.

Theorem 5. *When $p \neq a^2+1$ then $u_n(3) < u_n(2)$.*

Proof. The solution of our problem in the interval $(ps^2, p(s+1)^2)$ consists of a suitable u^2m and pv^2m where u/v is a convergent to $\sqrt{p}$ and m is a suitable integer depending on s.

Given any convergent u/v to $\sqrt{p}$ we define the *norm* to be $u^2 - pv^2$ and by the associated *pair* we will mean the pair (u^2, pv^2). The sequence of norms is periodic with each period ending with a convergent having norm 1.

Each convergent is used for a range of consecutive values of s, and the interval $(ps^2, p(s+1)^2)$ where we change from one convergent to the next is called a *critical* interval. The critical interval begins at the product of the smaller numbers of each pair and ends at the product of the larger numbers of each pair.

We illustrate our method with the prime 19 and the convergent 9/2.

We use the pair $(9^2, 19\cdot 2^2) = (81, 76)$ until we reach the critical interval where we change to the next convergent 13/3 with its pair $(13^2, 19\cdot 3^2) = (169, 171)$. This is the interval $(76\cdot 169, 81\cdot 171) = (19\cdot 26^2, 19\cdot 27^2)$. In this interval, note that if we use the pair $(81, 76)$, we need a multiplier strictly between 169 and 171. If we use $(169, 171)$ we have $76 < m < 81$.

Between $19\cdot 9^2$ and $19\cdot 26^2$ we use the pair $(81, 76)$ and we show that there is always a suitable integer m such that $19s^2 < 76m$ and $81m < 19(s+1)^2$. Given s, we can clearly satisfy these inequalities with integer m, provided $Q(s) = 19(s+1)^2/81 - 19s^2/76 > 1$.

The quadratic function $Q(s)$ enjoys $Q(8) = 3$ and $Q(26) = 2$. Considering the parabola $Q(s)$ and its derivatives it is clear that $Q(s) > 2$ for $8 < s < 26$. Thus for each value of $s \in (8, 26)$, the number of integers m is at least 2. Thus we have solved the problem when $p = 19$ for each interval between the two critical intervals involving the convergent 9/2.

If one of the pairs forming a critical interval has norm 1 or -1, then this pair must be used for the solution in that interval. For example, when $p = 7$, with convergents 5/2 and 8/3, the interval $(7(3\cdot 5)^2, 7(2\cdot 8)^2) = (25\cdot 63, 28\cdot 64)$ is critical. There is no integer m available for the pair $(25, 28)$, but the pair $(64, 63)$ may be used with an integer $25 < m < 28$.

This method works for every prime and every interval as long as p is not of the form a^2+1. Those are the primes for which each convergent has norm -1 or 1. □

Theorem 6. *When $p = a^2+1$ and $p > 2$, then $u_n(3) < u_n(2)$.*

Proof. In this case we use the pair $((u_{i+1} - u_i)^2, p(v_{i+1} - v_i)^2)$, to find a solution in the critical interval between the intervals where we use the pair (u_i^2, pv_i^2) and the pair

Table 3
Solutions in critical intervals $(2s^2, 2(s+1)^2)$ when $p=2$

s	$s+1$	Solution count	One solution
$1\cdot 2$	$1\cdot 3$	0	
$2\cdot 7$	$3\cdot 5$	0	
$7\cdot 12$	$5\cdot 17$	2	$45^2 7,\ 2\cdot 32^2 7$
$12\cdot 41$	$17\cdot 29$	1	$263^2 7,\ 2\cdot 186^2 7$
$41\cdot 70$	$29\cdot 99$	3	$2344^2 3,\ 2\cdot 1657^2 3$
$70\cdot 239$	$99\cdot 169$	4	$3527^2 45,\ 2\cdot 2494^2 45$
$239\cdot 408$	$169\cdot 577$	7	$1591^2 7513,\ 2\cdot 1125^2 7513$

(u_{i+1}^2, pv_{i+1}^2). First, we show that this works when $u_1=a$, $v_1=1$ and $u_2=2a^2+1$, $v_2=2a$. The critical interval is $(p(a\cdot 2a)^2, p(2a^2+1)^2)$. We show that there is an integer multiple m of $p(v_2-v_1)^2$ in the interval and since $m(u_2-u_1)^2=mp(v_2-v_1)^2+2am$, we need only show that $p(v_2-v_1)^2+2am<p(4a^2+1)$, which is the length of the interval. The multiplier must be greater than $4a^4/(2a-1)^2$. We use a^2+a+2, since $(a^2+a+2)(2a-1)^2>4a^4$ for $a>1$. We are done, since $(a^2+1)(2a-1)^2+2a(a^2+a+2)<(a^2+1)(4a^2+1)$.

In fact, when we increase the subscripts of the u's and v's there is relatively more room for $p(v_{i+1}-v_i)^2$ and $2am$ in $(p(u_iv_{i+1})^2, p(u_{i+1}v_i)^2)$. We note here that $u_{j+1}=2au_j+u_{j-1}$ and $v_{j+1}=2av_j+v_{j-1}$. □

When $p=2$, again the method above finds solutions for all but the critical intervals. We list in Table 3 the products, s and $s+1$, of entries of consecutive convergents to $\sqrt{2}$. Here the critical interval is $(2s^2, 2(s+1)^2)$. We list the number of solutions in the critical interval and show one solution when there is one. For example, 7/5 and 17/12 are consecutive convergents and the critical interval is $(2(7\cdot 12)^2, 2(5\cdot 17)^2)=(2\cdot 84^2, 2\cdot 85^2)$. There are two solutions in this interval, one of which is $(45^2\cdot 7,\ 2\cdot 32^2\cdot 7)$.

The reader can verify immediately that $u_8(3)=u_8(2)=18-8$. The next critical interval is $(2\cdot 14^2, 2\cdot 15^2)=(392,450)$. Try $32\cdot 13$, then $8\cdot 51$, $8\cdot 53$, $8\cdot 55$, and finally $2\cdot 197$, $2\cdot 199, \ldots, 2\cdot 223$. In each case we quickly see that the necessary companion is not in the interval, and there are no solutions for $n=392$.

Aaron Meyerowitz and Selfridge are planning to publish further results for the case $p=2$.

4. The gap size restriction

We discuss now the kth power problem with no gap greater than k.

Definition. For n not a kth power, let $g_k(n)$ be the minimum integer such that a kth power may be formed as the product of n and integers from the interval $(n, g_k(n)]$, so that no k consecutive integers are omitted.

Table 4
Primes whose doubles appear in *2-blocks*

2	41	173	281	439	619	761	1009	1237	1481	1733
3	53	179	293	443	641	809	1013	1279	1499	1759
5	79	191	307	491	653	811	1019	1289	1511	1811
7	83	199	331	499	659	829	1031	1297	1531	1867
11	89	211	337	509	661	877	1049	1399	1559	1889
19	97	229	359	547	683	911	1069	1409	1583	1901
23	113	233	367	577	691	937	1103	1429	1601	1931
29	131	239	379	593	719	953	1171	1439	1609	1973
31	139	251	419	601	727	967	1223	1451	1627	2003
37	157	271	431	607	743	997	1229	1459	1657	2011

In the case $k=2$, an easy proof shows that $g_2(n)<4n$. We modify this proof to show that for $n>12$, $g_2(n)\leqslant 3n-3$, and conjecture that infinitely often $3n-3$ is best possible. We conjecture a stronger result and provide numerical evidence. These investigations are based on the following concept of *blocks*.

Suppose we start with $n=8$ and wish to form a square with no gaps of length more than two. It is easy to see that we cannot form a square using 8 and larger numbers all smaller than 13. If we include 13 in our product, then we cannot hope to finish before obtaining another multiple of 13. On the other hand, to omit 13 necessitates the inclusion of 14. Thus we are forced to continue as far as the next multiple of 7. Indeed, $8\cdot 10\cdot 12\cdot 14\cdot 16\cdot 18\cdot 20\cdot 21$ is a square. We regard the pair $\{13,14\}$ as a *2-block*, (or simply *block*, when $k=2$ is understood), since it contains a prime adjacent to twice a prime. Similarly, for $k=3$, the blocks consist of 3 consecutive numbers $\{q_1,2q_2,3q_3\}$, with q_i prime. It is clear that the $2q_2$ must lie between the q_1 and $3q_3$.

Definition. A *k-block* is an interval of k numbers which is a permutation of the numbers iq_i, where the q_i are primes and $i=1,2,\ldots,k$.

Remark. The possible permutations in a k-block have been studied in [2].

For $k=2$ we conjecture the following improvement on the lower bound.

Conjecture 7. Let p be the largest prime $<n$ so that one of $2p\pm 1$ is also prime; then for $n>22$, $g_2(n)=3p$.

We investigate two conjectures (Conjectures 10, 11 below) that together would imply Conjecture 7. Roughly speaking, one says that the 2-blocks are close to one another, and the other that the interval $[n,g_2(n)]$ will always contain a 2-block. A list of primes $q<2029$ for which $2q$ is in a 2-block is given in Table 4.

Similarly, there are two more general conjectures (Conjectures 12, 13 below) for $k\geqslant 2$. One says that k-blocks are close to one another, and the other that for large n the interval $[n,g_k(n)]$ will always contain a k-block. For any given k, these conjectures immediately imply that for large n a kth power cannot be formed as the product of n and larger integers without omitting $k-1$ consecutive integers.

In particular, if $k=3$, and we try to form a cube without omitting two consecutive integers, Conjecture 13 using $k=3$ implies that for n large we must hit a 3-block. This 3-block will contain a 2-block, and one of these integers must be included in the product. Now Conjecture 10 applies, and we see that the process will never terminate. Looking at Table 4, we expect that no numbers greater than or equal to 37 may be included. We see immediately from the 2-block $\{22,23\}$ that the product must contain 11, 22, and 33, or else all numbers must be less than 22. The reader can complete the short list of solutions which begins with 2×4.

Results for small values of k and computational evidence suggest the general upper bound $g_k(n)\leqslant(k+1)(n-1)$. Again, corresponding to the case $k=2$, above, there is a stronger conjecture involving blocks.

Conjecture 8. Let p be the largest prime $<n$ such that kp appears in a k-block. Then $g_k(n)=(k+1)p$, when $n>N(k)$.

5. Squares with gaps $\leqslant 2$

The following short proof (by E. Szekeres) shows that

$$g_2(n)\leqslant\begin{cases}4n-4 & \text{if } n \text{ is odd,}\\ 4n-2 & \text{if } n \text{ is even.}\end{cases}$$

Take the numbers $n, n+1, \ldots, 2n-2$. These may be paired off with their doubles $2n, 2(n+1), \ldots, 2(2n-2)$. If the number of these pairs is even, the product of all these numbers will be a square. If not, add the pair $2n-1, 4n-2$.

With a similar but somewhat more complicated pairing argument, we show

Theorem 9. *For $n>6$, except $n=12$,*

$$g_2(n)\leqslant 3n-3.$$

Proof. The interval $[n,3n-3]$ will be subdivided into five parts. For notational convenience, let $m=n-1$.

Take every integer r_1 from $(m,9m/8)$, and also their doubles $2r_1$, from $(2m,9m/4)$. From $[9m/8,3m/2)$ take all integers not $\equiv 0 \pmod 3$, i.e., of the form $3r_2\pm 1$, and their doubles $6r_2\pm 2$ from $[9m/4,3m)$. Then take all even numbers $2r_3$ in $[3m/2,2m]$ and all $3r_3$ from $[9m/4,3m]$. The condition that no two consecutive numbers are omitted is clearly satisfied, since the largest integer less than $3m/2$ is not divisible by 3, and thus is always chosen in the second subinterval.

Now, we need only ensure that the parity of the 2's and of the 3's is even. The parity of the 3's can be changed if necessary using $3(2a+1)^2$, as follows. If $3(2a+1)^2$ is not

Table 5
Products for $g_2(n)$ for $2 \leqslant n \leqslant 22$

2	3 4 6 $(=4n-2)$
3	4 6 8 $(=4n-4)$
5	6 7 8 10 12 14 $(=4n-2$ with $n=4)$
6	7 9 10 12 14 16 18 20 $(=4n-2$, omitting 8, 11, 22)
7	9 10 12 14 15
8	10 12 14 16 18 20 21
10	11 12 14 16 18 20 21 22
11	12 14 15 16 18 20 21 22 24
12	13 14 16 17 18 20 22 24 26 28 30 32 33 34
13	14 16 17 18 20 22 24 26 27 28 30 32 33 34 (product for 12: $\times 27 \div 12$)
14	15 16 18 20 21 (6-pack!)
15	17 19 21 22 24 25 27 28 30 32 33 34 36 38
17	18 19 20 21 22 24 26 27 28 30 32 33 34 36 38 39
18	19 20 22 24 26 28 30 32 33 35 36 38 39 40
19	20 22 24 25 27 28 30 32 33 35 36 38 40 (18: $\times 25 \times 27 \div 18 \div 26 \div 39$)
20	21 22 23 25 26 28 30 32 34 36 38 40 42 44 45 46 48 50 51 52 54 56 57
21	22 23 25 26 28 $\ldots 2k \ldots$ 48 50 51 52 54 56 57 (20: $\div 20 \div 45$)
22	24 25 27 28 30 32 33 35

already included in the product, include it. If $3(2a+1)^2$ has been included, then the adjacent integers, $3(2a+1)^2 \pm 1$, have also been included, and so we delete $3(2a+1)^2$ from our list. This can be done when $n \geqslant 10$.

But first, if necessary, correct the parity of the 2's. This is accomplished by the following, which also changes the parity of the 3's. Select a number $6b$ which is $6 \bmod 12$ in $(n, 3m/2)$. This can be done provided $n > 21$. Add $9b$, and if $6b \geqslant 9m/8$, add $6b$, otherwise delete $6b$ (whose neighbors are both already in). Table 5 contains the solutions for $n \leqslant 22$. Notice that $g_2(n) \leqslant 3n$, except when $n = 6$. □

Remark. With p as defined in Conjecture 7 above, if $n = p+1$, then Theorem 9 shows that $g_2(n) \leqslant 3p$. So $g_2(n) = 3n - 3$ infinitely often is implied by the two conjectures given below.

The connection between blocks and $g_2(n) \geqslant 3p$ is illustrated by the following examples.

Example. $g_2(23) = 57$. Starting with 23, we must go as far as 46 to get another multiple of 23. We must use 37 or 38, so cannot finish before 57. So $g_2(23) \geqslant 57$. Equality is established by the product of $23 \cdot 27 \cdot 35 \cdot 51 \cdot 55 \cdot 57$ and all evens between 23 and 57.

Example. $g_2(24) = 69$. First note that we cannot finish before 27, and so must use 28 or 29. Continuing, we must then pass the block 33, 34, and hence go at least as far as 44, passing 37, 38. Then $g_2(24) \geqslant 57$, and the block 46, $47 \Rightarrow g_2(24) \geqslant 69$ $(=3n-3)$. (Equality follows from Theorem 9).

Example. $g_2(32) = 93$. The block 33, 34 $\Rightarrow g_2(32) \geqslant 44$, then 37, 38 $\Rightarrow \geqslant 57$, 46, 47 $\Rightarrow \geqslant 69$, and finally, 61, 62 $\Rightarrow \geqslant 93$ $(= 3n - 3)$.

Example. It is immediate from 106, 107 that $g_2(79) \geqslant 159$. A lengthy hand computation verifies that $g_2(79) = 159$.

Example. $g_2(80) = 237$. Since $g_2(80) \geqslant 85$, the block 82, 83 $\Rightarrow \geqslant 123$, and 106, 107 $\Rightarrow \geqslant 159$, then 157, 158 $\Rightarrow \geqslant 237$ $(= 3n - 3)$.

The conjectured lower bound $g_2(n) \geqslant 3p$ would follow from the conjunction of the following two conjectures.

Conjecture 10. The ratio of primes in successive 2-blocks is less than $1 + \varepsilon$ for blocks $> N(\varepsilon)$.

In particular we note from Table 4 that this ratio seems to be $< 3/2$ beyond 22, 23.

Conjecture 11. For $n > 22$, the interval $(n, g_2(n)]$ contains a 2-block.

Conjecture 11 has been verified by computations using *Mathematica* up to 5×10^5. Similar results seem to hold for other small values of k, and provide some support for the following more general conjectures corresponding to Conjectures 10 and 11.

Conjecture 12. Given k, the ratio of primes in successive k-blocks is less than $1 + \varepsilon$ for k-blocks $> N(k, \varepsilon)$.

Conjecture 13. For n sufficiently large, and in particular greater than the first k-block with all larger k-block ratios less than $1 + 1/k$, the interval $(n, g_k(n)]$ contains a k-block.

Correspondingly, the lower bound $g_k \geqslant (k + 1)p$ in Conjecture 8 is implied by the conjunction of Conjectures 12 and 13.

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